Convergence rate of EM algorithm for SDEs under integrability condition

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July 15, 2019

Outline

- Motivations
- Convergence rate of EM scheme with integrable drift
- Convergence rate of EM scheme with non-integrable drift
- Proofs of main results

This talk is based on a joint paper with Xing Huang & Shaoqin Zhang.

Motivations

- Strong/weak convergence of numerical schemes for SDEs with regular coefficients has been investigated extensively; Kloeden-Platen (1992);
- EM scheme is valid merely to SDEs with linear growth; Jentzen, et al. (CMS, 2016);
- Backward EM scheme (Higham, et al., 2002), tamed EM scheme (Dareiotis, et al., 2016; Sabanis, 2016), truncated EM scheme (Mao, 2015), $\cdots$;

Motivations

Convergence of numerical algorithms for SDEs with irregular coefficients:

- Gyöngy-Rásonyi (SPA, 2011): SDEs with Hölder continuous diffusions via Yamada-Watanabe approach;
- Yan (AOP, 2002): SDEs with Hölder continuous drifts by Meyer-Tanaka formula & estimates on local times;
- Pamen-Taguchi (SPA, 2017); SDEs with Hölder continuous drift using the regularity of backward Kolmogrov equations (i.e., Zvonkin's transformation);

Numerical approximations of SDEs with discontinuous drifts:

- Leobacher-Szölgyenyi (AAP, 2017) & Leobacher-Szölgyenyi (Numer. Math., 2018) under piecewise Lipschitz assumption.

Motivations

SDEs with integrable coefficients:

- Zvonkin (1974): pioneer work;
- Krylov-Röckner (PTRF, 2005): additive noise;
- Zhang (SPA, 2005; EJP, 2011): multiplicative noise;
- Huang-Wang (SPA, 2019) & Röckner-Zhang (2018): McKean-Vlasov SDEs;
- $\cdots$
-

Approximations of SDEs with integrable coefficients:

- Ngo-Taguchi (2016): integrability condition & one-sided Lipschitz;
- Ngo-Taguchi (2017): integrability condition & 1-dimension.

Framework

In this talk, we consider

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \quad t \geq 0, \quad X_0 = x, \tag{1}$$

where

- $b : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$;
- $(W_t)_{t\geq 0}$ is a d-dimensional Brownian motion.

EM scheme associated with (1): For any $\delta \in (0, 1)$,

$$\mathrm{d}X_t^{(\delta)} = b(X_{t_\delta}^{(\delta)})\mathrm{d}t + \sigma(X_{t_\delta}^{(\delta)})\mathrm{d}W_t, \quad t \geq 0, \quad X_0^{(\delta)} = X_0, \tag{2}$$

where $t_\delta := \lfloor t/\delta \rfloor \delta$.

Assumptions on Coefficients

(A1) $\|b\|_\infty := \sup_{x\in\mathbb{R}^d} |b(x)| < \infty$ and there is $p > \frac{d}{2}$ such that $|b|^2 \in L^p$;

(A2) There exist a constant $\alpha > 0$ and a locally integrable function $\phi \in C(\mathbb{R}_+;\mathbb{R}_+)$ such that

$$\frac{1}{s^{\frac{d}{2}}}\int_{\mathbb{R}^d} |b(x+y)-b(x+z)|^2 \mathrm{e}^{-\frac{1}{s}|x|^2}\mathrm{d}x \le \phi(s)|y-z|^\alpha,\ y,z\in\mathbb{R}^d,\ s>0;$$

(A3) There exist constants $\breve{\lambda}_0, \widehat{\lambda}_0, L_0 > 0$ such that

$$\breve{\lambda}_0|\xi|^2 \le \langle(\sigma\sigma^*)(x)\xi,\xi\rangle \le \widehat{\lambda}_0|\xi|^2,\ x,\xi\in\mathbb{R}^d \qquad (UE) \qquad (3)$$

and

$$\|\sigma(x)-\sigma(y)\|_{\mathrm{HS}} \le L_0|x-y|, \quad x,y\in\mathbb{R}^d. \qquad (4)$$

Remarks

- Under (**A1**) and (**A3**), has a unique strong solution;
- $(X_{k\delta}^{(\delta)})_{k\geq 0}$ is a homogeneous Markov chain;
- (**A2**) is imposed to reveal the convergence rate of EM scheme;
- $b(x) = \mathbb{1}_{[a,b]}(x)$ satisfies (**A2**).

Main Result I

Theorem

Assume **(A1)-(A3)**. *Then, for* $\beta \in (0,2)$ *and* $q > 2$, *there exist constants* $C_1, C_2 > 0$ *such that*

$$\mathbb{E}\Big(\sup_{0\le t\le T} |X_t - X_t^{(\delta)}|^\beta\Big) \le C_1 \mathrm{e}^{C_2(1+\||b|^2\|_{L^p}^{\gamma_0})}\delta^{\frac{\beta}{2}(1\wedge\frac{\alpha}{2})}, \tag{5}$$

where $\gamma_0 := \frac{1}{1-1/q-d/2p}$.

Remarks:

- Investigate rate of EM scheme for SDEs with integrability condition;
- Drop the piecewise Lipschitz continuity on the drifts;
- Extend to high dimensional setting;

Further Remarks

(A2) can be replaced by **(A2')** below

(A2') There exist constants $\beta, \theta > 0$ such that

$$\frac{1}{(rs)^{d/2}} \sup_{z\in\mathbb{R}^d} \int_{\mathbb{R}^d\times\mathbb{R}^d} |b(x)-b(y)|^2 \mathrm{e}^{-\frac{|x-z|^2}{s}} \mathrm{e}^{-\frac{|y-x|^2}{r}} \mathrm{d}y\mathrm{d}x \le C r^\theta s^{\beta-1}$$

for some constant $C > 0$.

- The drift b satisfying **(A2')** is said to the Gaussian-Besov class with the index (β, θ), denoted by $GB^2_{\beta,\theta}(\mathbb{R}^d)$.
- For $\theta > 0$ and $p \in [2, \infty) \cap (d, \infty)$, if the Gagliardo seminorm

$$[b]_{W^{p,\theta}} := \Big(\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|b(x)-b(y)|^p}{|x-y|^{d+p\theta}} \mathrm{d}x\mathrm{d}y\Big)^{\frac{1}{p}} < \infty,$$

then $b \in GB^2_{1-\frac{d}{p},\theta}(\mathbb{R}^d)$.

Heat Kernel for Exact Solution

Lemma

Under $\|b\|_\infty < \infty$ *and* $(\mathbf{A3})$, *the transition density* p *of* $(X_t)_{t\geq s}$ *satisfies*

$$p(s,t,x,x') \leq \mathrm{e}^{\frac{\|b\|_\infty^2 T}{2\hat{\lambda}_0}} \sum_{i=0}^{\infty} \frac{\beta_T^i}{\Gamma(1+\frac{i}{2})} p_0(t-s,x,x'), \quad 0 \leq s < t \leq T, \tag{6}$$

where $\Gamma(\cdot)$ *is the Gamma function, and*

$$p_0(t,x,x') := \frac{\mathrm{e}^{-\frac{|x-x'|^2}{16\hat{\lambda}_0 t}}}{(2\pi\check{\lambda}_0 t)^{d/2}}. \tag{7}$$

- The proof is due to parametrix method (Konakov-Mammen, PTRF,2000);
- β_T is explicit.

Heat Kernel for Numerical Scheme

Lemma

Under $(\mathbf{A1})$ *and* $(\mathbf{A3})$,

$$p^{(\delta)}(j\delta,t,x,y)\le\frac{\Lambda_3 e^{-\frac{|y-x|^2}{\kappa_0(t-j\delta)}}}{(2\pi\breve{\lambda}_0(t-j\delta))^{d/2}},\quad x,y\in\mathbb{R}^d,\ \ t>j\delta,\ \ \delta\in(0,1).\tag{8}$$

Ideas for the proof

- parametrix method (Konakov-Mammen, PTRF,2000)+C-K equation;
- Λ_3 and κ_0 are explicit.

Estimate on Displacement of b

Lemma

Under **(A1)-(A3)**, *for any $T > 0$, there exists a constant $C > 0$ such that*

$$\int_0^T \mathbb{E}|b(X_t^{(\delta)}) - b(X_{t_\delta}^{(\delta)})|^2 \mathrm{d}t \le C\delta^{1\wedge\frac{\alpha}{2}}. \tag{9}$$

- It is easy to obtain (9) whenever b is Hölder continuity;
- The proof of (9) is based on Heat kernel estimate for numerical scheme;
- (9) is used to reveal the convergence rate of EM scheme.

Krylov's Estimate for Numerical Scheme

Lemma

Assume $(\mathbf{A1})$ *and* $(\mathbf{A3})$. *Then, for* $f \in L_q^p(T)$ *with* $(p,q) \in \mathscr{K}_1$,

$$\mathbb{E}\left(\int_s^t |f_r(X_r^{(\delta)})| \mathrm{d}r \middle| \mathscr{F}_s \right) \le \alpha_0 \, \|f\|_{L_q^p(T)} (t-s)^{\frac{1}{\gamma_0}}, \quad 0 \le s \le t \le T, \quad (10)$$

where $\gamma_0 := \frac{1}{1-1/q-d/2p}$.

- α_0 *is explicit;*
- *The proof of* (10) *is due to heat kernel estimate of numerical scheme.*

Khasminskill's Estimate for Numerical Scheme

Lemma

Assume $(\mathbf{A1})$ *and* $(\mathbf{A3})$. *Then, for* $f \in L_q^p(T)$ *with* $(p,q) \in \mathscr{K}_1$, *the following Khasminskill type estimate*

$$\mathbb{E}\exp\left(\lambda\int_0^T |f_t(X_t^{(\delta)})|\mathrm{d}t\right) \le 2^{1+T(2\lambda\alpha_0\|f\|_{L_q^p(T)})^{\gamma_0}}, \qquad \lambda > 0 \tag{11}$$

holds, where $\gamma_0 := \frac{1}{1-1/q-d/2p}$ *and*

$$\alpha_0 := \frac{(1-1/p)^{\frac{d}{2}(1-1/p)}}{(\breve{\lambda}_0(2\pi)^{\frac{1}{p}})^{\frac{d}{2}}}\left\{\widehat{\lambda}_0^{\frac{d}{2}(1-1/p)} + \Lambda_3(\gamma_0(1-1/q))^{\frac{q-1}{q}}(\kappa_0/2)^{\frac{d}{2}(1-1/p)}\right\}.$$

Remark: Follow the line of Xie & Zhang (2017).

Remark (Shao, J., 2018)

- The Krylov estimate associated with $(X_{t_\delta}^{(\delta)})_{t\geq 0}$ no longer holds true.
- Take $s, t \in [k\delta, (k+1)\delta)$ for some integer $k \geq 1$. Then,

$$\mathbb{E}\Big(\int_s^t |f_{r_\delta}(X_{r_\delta}^{(\delta)})|\mathrm{d}r\Big|\mathscr{F}_s\Big) = |f_{k\delta}(X_{k\delta}^{(\delta)})|(t-s) \tag{12}$$

which is a random variable. Hence, it is impossible to control the quantity on the left hand side of (12) by $\|f\|_{L_q^p(T)}$ up to a constant.

Regularity of Kolmogrov Backward Equation

For any $\lambda > 0$, consider the following PDE for $u^\lambda : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$:

$$\partial_t u^\lambda + \frac{1}{2} \sum_{i,j=1}^{d} \langle \sigma\sigma^* e_i, e_j \rangle \nabla_{e_i} \nabla_{e_j} u^\lambda + b + \nabla_b u^\lambda = \lambda u^\lambda. \tag{13}$$

- (13) has a unique solution $u^\lambda \in \mathscr{H}_{2p}^{2,2q}(0,T)$ satisfying

$$(1 \vee \lambda)^{1-\frac{d}{4p}-\frac{1}{2q}} \|\nabla u^\lambda\|_{T,\infty} + \|\nabla^2 u^\lambda\|_{L_{2q}^{2p}(T)} \le C_1 \||b|^2\|_{L^p} \tag{14}$$

  for some constant $C_1 > 0$; see e.g. Xie-Zhang (2017).
- There is a constant $\lambda_0 \ge 1$ such that

$$\|\nabla u^\lambda\|_{T,\infty} \le \frac{1}{2}, \quad \lambda \ge \lambda_0. \tag{15}$$

- We can apply Itô's formula to u by a standard approximation argument.

Key Points for the Proof of Main Result I

- For $\theta_t^\lambda(x) := x + u_t^\lambda(x), x \in \mathbb{R}^d$, and $Z_t^{(\delta)} := X_t - X_t^{(\delta)}$, by Itô's formula,

$$
\begin{aligned}
\mathrm{d}\theta_t^\lambda(X_t) &= \lambda u^\lambda(X_t)\mathrm{d}t + \nabla\theta_t^\lambda(X_t)\sigma(X_t)\mathrm{d}W_t \\
\mathrm{d}\theta_t^\lambda(X_t^{(\delta)}) &= \Big\{\lambda u^\lambda(X_t^{(\delta)}) + \nabla\theta_t^\lambda(X_t^{(\delta)})(b(X_{t_\delta}^{(\delta)}) - b(X_t^{(\delta)})) \\
&\quad + \frac{1}{2}\sum_{i,j=1}^d \langle(((\sigma\sigma^*)(X_{t_\delta}^{(\delta)}) \\
&\quad - (\sigma\sigma^*)(X_t^{(\delta)}))e_i, e_j\rangle\nabla_{e_i}\nabla_{e_j}u_t^\lambda(X_t^\delta))\Big\}\mathrm{d}t \\
&\quad + \nabla\theta_t^\lambda(X_t^{(\delta)})\sigma(X_{t_\delta}^{(\delta)})\mathrm{d}W_t.
\end{aligned}
$$

Key Point: Choose the same θ_t^λ.

Key Points for the Proof of Main Result I (Cont.)

- Again, by Itô's formula,

$$
\begin{aligned}
|Z_t^{(\delta)}|^2 \le\; & 8\lambda \int_0^t \langle \theta_s^\lambda(X_s) - \theta_s^\lambda(X_s^{(\delta)}), u^\lambda(X_s) - u^\lambda(X_s^{(\delta)})\rangle \mathrm{d}s \\
& + 8\int_0^t \langle \theta_s^\lambda(X_s) - \theta_s^\lambda(X_s^{(\delta)}), \nabla\theta_s^\lambda(X_s^{(\delta)})(b(X_s^{(\delta)}) - b(X_{s_\delta}^{(\delta)}))\rangle \mathrm{d}s \\
& + \sum_{i,j=1}^d \int_0^t \langle(((\sigma\sigma^*)(X_{s_\delta}^{(\delta)}) - (\sigma\sigma^*)(X_s^{(\delta)}))e_i, e_j\rangle \\
& \times \langle \theta_s^\lambda(X_s) - \theta_s^\lambda(X_s^{(\delta)}), \nabla_{e_i}\nabla_{e_j} u_s^\lambda(X_s^\delta))\rangle \mathrm{d}s \\
& + 4\int_0^t \|\nabla\theta_s^\lambda(X_s)\sigma(X_s) - \nabla\theta_s^\lambda(X_s^{(\delta)})\sigma(X_{s_\delta}^{(\delta)})\|_{\mathrm{HS}}^2 \mathrm{d}s + M_t \\
=:\; & I_{1,\delta}(t) + I_{2,\delta}(t) + I_{3,\delta}(t) + I_{4,\delta}(t) + M_t.
\end{aligned}
$$

Key Points for the Proof of Main Result I (Cont.)

- Handle $I_{1,\delta}(t)$ via (15);
- Deal with $I_{2,\delta}(t)$ by (15) & Lemma 4;
- Cope with $I_{3,\delta}(t)$ by (**A3**) & (15).
- Estimate $I_{4,\delta}(t)$ by the Hardy-Littlewood maximum theorem;
- Obtain the estimate

$$\Big(\mathbb{E}\Big(\sup_{0\le s\le t}|Z_t^{(\delta)}|^{2\kappa'}\Big)\Big)^{1/\kappa'}, \quad \kappa'\in(0,1)$$

by stochastic Gronwall inequality (Xie-Zhang, 2017) and Khasminskill's estimate.

Main Result II

- In Theorem 8, the integrable condition (i.e., $|b|^2 \in L^p$) seems to be a little bit restrictive;
- It rules out some typical examples, e.g., $b(x) = \mathbf{1}_{[0,\infty)}(x)$;
- By implementing a truncation argument, the integrable condition can indeed be dropped.

Theorem

Assume **(A1)-(A3)** *without* $|b|^2 \in L^p$. *Then, for* $\beta \in (0,2)$, *and* $p, q > 2$ *with* $\frac{d}{p} + \frac{1}{q} < 1$, *there exist constants* $C_1, C_2 > 0$ *such that*

$$\mathbb{E}\Big(\sup_{0\le t\le T} |X_t - X_t^{(\delta)}|^\beta\Big) \le C_1\Big\{\mathrm{e}^{C_2(-\frac{\beta}{2}(1\wedge\frac{\alpha}{2})\log\delta)^{\frac{d\gamma_0}{2p}}} + 1\Big\}\delta^{\frac{\beta}{2}(1\wedge\frac{\alpha}{2})}. \quad (16)$$

Key Points for the Proof of Main Result II

- Let $\psi : \mathbb{R}_+ \to [0,1]$ be a smooth function such that $\psi(r) = 1, r \in [0,1]$, and $\psi(r) \equiv 0, r \geq 2$.
- For each integer $k \geq 1$, let $b_k(x) = b(x)\psi(|x|/k)$, $x \in \mathbb{R}^d$, which satisfies

$$\|b_k\|_\infty \leq \|b\|_\infty \quad \text{and} \quad \||b_k|^2\|_{L^p} \leq \Big(\frac{2^d \pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}\Big)^{1/p} k^{\frac{d}{p}} \|b\|_\infty^2. \quad (17)$$

- Consider the following truncated SDE corresponding to (1)

$$\mathrm{d}X_t^k = b_k(X_t^k)\mathrm{d}t + \sigma(X_t^k)\mathrm{d}W_t, \quad t \geq 0, \quad X_0^k = X_0. \quad (18)$$

Key Points for the Proof of Main Result II (Cont.)

- For $q \in (0,2)$, observe that

$$\mathbb{E}\|X - X^{(\delta)}\|_{T,\infty}^q \le 3^{0\vee(q-1)}\{\mathbb{E}\|X - X^k\|_{T,\infty}^q + \mathbb{E}\|X^{(\delta)} - X^{k,(\delta)}\|_{T,\infty}^q + \mathbb{E}\|X_t^k - X^{k,(\delta)}\|_{T,\infty}^q\} =: 3^{0\vee(q-1)}\{I_1 + I_2 + I_3\}.$$

- According to martingale inequality (Shigekawa, 2004),

$$I_1, I_2 \le C_2 \exp\Big(\frac{(|x| + \|b\|_\infty T)^2}{8d^2\widehat{\lambda}_0 T}\Big)\mathrm{e}^{-\frac{k^2}{16d^2\widehat{\lambda}_0 T}} \tag{19}$$

- By virtue of Main Result I,

$$I_3 \le C_4 \mathrm{e}^{C_6\|b\|_\infty^{2\gamma_0} k^{\frac{d\gamma_0}{p}}} \delta^{\frac{q}{2}(1\wedge\frac{\alpha}{2})} \tag{20}$$

- The desired assertion is available by taking

$$k = \Big(-8qd^2\widehat{\lambda}_0 T\Big(1\wedge\frac{\alpha}{2}\Big)\log\delta\Big)^{\frac{1}{2}}.$$

Summaries

Theorem

Assume **(A1)-(A3)**. *Then, for* $\beta \in (0,2)$ *and* $q > 2$, *there exist constants* $C_1, C_2 > 0$ *such that, for* $\gamma_0 := \frac{1}{1-1/q-d/2p}$,

$$\mathbb{E}\Big(\sup_{0\le t\le T} |X_t - X_t^{(\delta)}|^\beta\Big) \le C_1 \mathrm{e}^{C_2(1+\||b|^2\|_{L^p}^{\gamma_0})}\delta^{\frac{\beta}{2}(1\wedge\frac{\alpha}{2})}. \tag{21}$$

Theorem

Assume **(A1)-(A3)** *without* $|b|^2 \in L^p$. *Then, for* $\beta \in (0,2)$, *and* $p, q > 2$ *with* $\frac{d}{p} + \frac{1}{q} < 1$, *there exist constants* $C_1, C_2 > 0$ *such that*

$$\mathbb{E}\Big(\sup_{0\le t\le T} |X_t - X_t^{(\delta)}|^\beta\Big) \le C_1 \Big\{\mathrm{e}^{C_2(-\frac{\beta}{2}(1\wedge\frac{\alpha}{2})\log\delta)^{\frac{d\gamma_0}{2p}}} + 1\Big\}\delta^{\frac{\beta}{2}(1\wedge\frac{\alpha}{2})}. \tag{22}$$

Thanks A Lot !